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An Efficient Numerical Method for Compressible Flows of a Real Gas Using Arithmetic Averaging

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Abstract—An efficient numerical method is presented for the solution of the Euler equations governing the compressible flow of a real gas. The scheme is based on the approximate solution of a specially constructed set of linearised Riemann problems. An average of the flow variables across the interface between cells is required, and this is chosen to be the arithmetic mean for computational efficiency, which is in contrast to the usual square root averaging. The scheme is applied to a test problem for five different equations of state.

1. INTRODUCTION

In a recent paper [1], an efficient numerical scheme was presented for the Euler equations for the compressible flow of an ideal gas. The essential difference between the scheme presented in [1] and similar schemes of this type is in the use of the arithmetic mean for the average of the flow variables across the interface between adjacent computational cells and this gives rise to an improvement in efficiency. The purpose of this paper is to extend this scheme to apply to flows of a real gas, i.e., ones where there is a nonideal equation of state. The aim is to utilise arithmetic averages again and, although the derivation is more detailed than the ideal case, the implementation is equally straightforward, with a corresponding improvement in the efficiency over similar schemes. The scheme is shown to coincide with that in [1] for the ideal gas case. Finally, the scheme is applied to a test problem involving shock reflection for five equations of state and a comparison is made with the exact solution.

2. GOVERNING EQUATIONS

The equations governing the compressible flow of a real gas can be written in conservation form as:

$$\mathbf{w}_t + \mathbf{F}_x = \mathbf{0}, \tag{2.1}$$

where

$$\mathbf{w} = (\rho,\, \rho u,\, e)^\top, \tag{2.2a}$$

$$\mathbf{F}(\mathbf{w}) = \left(\rho u,\, p + \rho u^2,\, u\,(e + p)\right)^\top, \tag{2.2b}$$

where

$$p = \rho i + \frac{1}{2}\rho u^2. \tag{2.3}$$

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The quantities ρ, u, p, i, and e represent the density, velocity, pressure, specific internal energy, and the total energy, respectively, at a general position x and at time t. In addition, there is an equation of state of the form

$$p = p(\rho, i) \tag{2.4}$$

relating p, ρ, and i. For an ideal gas, equation (2.4) becomes

$$p = (\gamma - 1)\,\rho\, i, \tag{2.5}$$

where the constant γ is the ratio of specific heat capacities of the gas.

3. APPROXIMATE RIEMANN SOLVER

We consider the approximate solution $\mathbf{w}_j^n \simeq \mathbf{w}(x_j,\, t_n)$ to consist of a set of piecewise constants, and solve approximately the associated Riemann problems at the interface separating adjacent states. An approximate Jacobian needs to be constructed across an interface so that shock capturing is automatic, and this represents an average of the Jacobian matrix evaluated at either side of the interface.

3.1. Structure

The Jacobian matrix

$$\mathbf{A} = \begin{pmatrix} 0 & 1 & 0 \\ a^2 - u^2 \dfrac{p_i}{\rho}\left(\dfrac{p}{\rho} + i - \dfrac{1}{2}u^2\right) & 2u - u\,\dfrac{p_i}{\rho} & \dfrac{p_i}{\rho} \\ u\,a^2 - u\left(\dfrac{p}{\rho} + i + \dfrac{1}{2}u^2\right) - u\,\dfrac{p_i}{\rho}\left(\dfrac{p}{\rho} + i - \dfrac{1}{2}u^2\right) & \dfrac{p}{\rho} + i + \dfrac{1}{2}u^2 - u^2\,\dfrac{p_i}{\rho} & u + u\,\dfrac{p_i}{\rho} \end{pmatrix} \tag{3.1}$$

of the flux function $\mathbf{F}(\mathbf{w})$ has eigenvalues λ_i, with corresponding eigenvectors $\mathbf{e}_i$, $i = 1,\, 2,\, 3$, given by

$$\lambda_1 = u + a, \quad \mathbf{e}_i = \left(1,\, u + a,\, \frac{p}{\rho} + i + \frac{1}{2}u^2 + u\,a\right)^{\top}, \tag{3.2a}$$

$$\lambda_2 = u - a, \quad \mathbf{e}_2 = \left(1,\, u - a,\, \frac{p}{\rho} + i + \frac{1}{2}u^2 - u\,a\right)^{\top}, \tag{3.2b}$$

$$\lambda_3 = u, \qquad \mathbf{e}_3 = \left(1,\, u,\, i + \frac{1}{2}u^2 - \rho\,\frac{p_\rho}{p_i}\right)^{\top}. \tag{3.2c}$$

The sound speed, a, is given by

$$a^2 = p_\rho + p\,\frac{p_i}{\rho^2}, \tag{3.3}$$

using the derivatives of the equation of state (2.4).

3.2. Shock Capturing

Consider two adjacent states $\mathbf{w}_L$, $\mathbf{w}_R$ (left and right) given at either end of the cell $(x_L,\, x_R)$, and consider also the algebraic problem of finding an approximate Jacobian $\tilde{\mathbf{A}} = \tilde{\mathbf{A}}(\mathbf{w}_L, \mathbf{w}_R)$ in this cell, such that

$$\tilde{\mathbf{A}}\;\Delta\mathbf{w} = \Delta\mathbf{F}, \tag{3.4}$$

where $\Delta(\cdot) = (\cdot)_R - (\cdot)_L$, $\mathbf{w} = (\rho,\, \rho u,\, e)^{\top}$, and $\mathbf{F} = (\rho u,\, p + \rho u^2, u\,(e + p))^{\top}$. A solution to this problem, for arbitrary jumps $\Delta\mathbf{w}$, can be used to obtain a conservative scheme with good shock-capturing properties.

3.3. Construction of $\tilde{\mathbf{A}}$

As in [1] for the ideal gas case, where much simplification arises due to the equation of state for an ideal gas (2.5), to determine the matrix $\tilde{\mathbf{A}}$, we first write $\Delta\mathbf{w}$ and $\Delta\mathbf{F}$ in terms of $\Delta\mathbf{u}$, where $\mathbf{u} = (\rho,\, u,\, i)^{\mathsf{T}}$. We note, however, that instead of the pressure, p, the third component of $\mathbf{u}$ is the specific internal energy, i. Following the identities

$$\Delta\rho = \Delta\rho, \tag{3.5}$$

$$\Delta(\rho u) = \bar\rho\,\Delta u + \bar u\,\Delta\rho, \tag{3.6}$$

$$\Delta(\rho i) = \bar\rho\,\Delta i + \bar i\,\Delta\rho, \qquad \text{and} \tag{3.7}$$

$$\Delta(\rho u^2) = \overline{u^2}\,\Delta\rho + \bar\rho\,\Delta u^2 = \overline{u^2}\,\Delta\rho + 2\bar\rho\,\bar u\,\Delta u, \tag{3.8}$$

where

$$\bar\rho = \frac{1}{2}\left(\rho_L + \rho_R\right), \tag{3.9}$$

$$\bar u = \frac{1}{2}\left(u_L + u_R\right), \qquad \text{and} \tag{3.10}$$

$$\bar i = \frac{1}{2}\left(i_L + i_R\right), \tag{3.11}$$

the arithmetic mean of left and right states, and

$$\overline{u^2} = \frac{1}{2}\left(u_L^2 + u_R^2\right), \tag{3.12}$$

the arithmetic mean of the square of the velocity, we can write

$$\Delta\mathbf{w} = \tilde{\mathbf{B}}\,\Delta\mathbf{u}, \tag{3.13}$$

where

$$\tilde{\mathbf{B}} = \begin{pmatrix} 1 & 0 & 0 \\ \bar u & \bar\rho & 0 \\ \bar i + \frac{1}{2}\overline{u^2} & \bar\rho\,\bar u & \bar\rho \end{pmatrix}. \tag{3.14}$$

Similarly,

$$\Delta p = \tilde p_\rho\,\Delta\rho + \tilde p_i\,\Delta i, \tag{3.15}$$

where $\tilde p_\rho$ and $\tilde p_i$ are appropriate averages of the derivatives of the equation of state chosen to satisfy this equation exactly (see Section 3.6);

$$\Delta(up) = \bar u\,\Delta p + \bar p\,\Delta u, \tag{3.16}$$

where

$$\bar p = \frac{1}{2}\left(p_L + p_R\right), \qquad \text{and} \tag{3.17}$$

$$\Delta\left(\rho i u\right) = \overline{\rho i}\,\Delta u + \bar u\,\Delta(\rho i) = \overline{\rho i}\,\Delta u + \bar u\,\bar\rho\,i + \bar u\bar i\,\Delta\rho, \tag{3.18}$$

where

$$\overline{\rho i} = \frac{1}{2}\left(\rho_L\, i_L + \rho_R\, i_R\right). \tag{3.19}$$

Finally,

$$\Delta\left(\rho u^3\right) = \overline{u^3}\,\Delta\rho + \bar p\,\Delta u^3 = \overline{u^3}\,\Delta\rho + 3\bar p\,\widetilde{u^2}\,\Delta u, \tag{3.20}$$

where

$$\overline{u^3} = \frac{1}{2}\left(u_L^3 + u_R^3\right), \qquad \text{and} \tag{3.21}$$

$$\widetilde{u^2} = \frac{1}{3}\left(u_L^2 + u_L\,u_R + u_R^2\right) \tag{3.22}$$

is an average of the square of the velocity. (N.B. The choice in equation (3.18) is made so that the eigenvalues of the approximate Jacobian $\tilde{\mathbf{A}}$ have the simplest possible form. In particular, $\tilde{\mathbf{A}}$ has $\bar{u}$ as one eigenvalue.) Combining equations (3.6), (3.8), (3.15), (3.16), (3.18), and (3.20) gives

$$\Delta\mathbf{F} = \tilde{\mathbf{C}}\,\Delta\mathbf{u}, \tag{3.23}$$

where

$$\tilde{\mathbf{C}} = \begin{pmatrix} \bar{u} & \bar{\rho} & 0 \\ \tilde{p}_\rho + \overline{u^2} & 2\bar{\rho}\,\bar{u} & \tilde{p}_i \\ \bar{u}\,\tilde{p}_\rho + \bar{u}\,\bar{i} + \dfrac{\overline{u^3}}{2} & \bar{p} + \dfrac{3\bar{\rho}\,\widetilde{u^2}}{2} + \overline{\rho i} & \bar{u}\,\tilde{p}_i + \bar{\rho}\,\bar{u} \end{pmatrix}, \tag{3.24}$$

and thus, from equations (3.13) and (3.23),

$$\Delta\mathbf{F} = \tilde{\mathbf{C}}\,\tilde{\mathbf{B}}^{-1}\,\Delta\mathbf{w}. \tag{3.25}$$

Therefore, a solution of equation (3.41) is obtained with the approximate Jacobian

$$\tilde{\mathbf{A}} = \tilde{\mathbf{C}}\,\tilde{\mathbf{B}}^{-1} = \begin{pmatrix} 0 & 1 & 0 \\ \widetilde{a^2} - \hat{u}^2 - \dfrac{\tilde{p}_i}{\bar{\rho}}\left(\dfrac{\bar{p}}{\bar{\rho}} + \bar{i} - \dfrac{1}{2}\,\hat{u}^2\right) & 2\bar{u} - \bar{u}\,\dfrac{\tilde{p}_i}{\bar{\rho}} & \dfrac{\tilde{p}_i}{\bar{\rho}} \\ \bar{u}\,\widetilde{a^2} - \bar{u}\left(\dfrac{\bar{p}}{\bar{\rho}} + \dfrac{\overline{\rho i}}{\bar{\rho}} + \dfrac{1}{2}\,\hat{u}^2\right) - \bar{u}\,\dfrac{\tilde{p}_i}{\bar{\rho}}\left(\dfrac{\bar{p}}{\bar{\rho}} + \bar{i} - \dfrac{1}{2}\,\hat{u}^2\right) & \dfrac{\bar{p}}{\bar{\rho}} + \dfrac{\overline{\rho i}}{\bar{\rho}} + \dfrac{\overline{u^2}}{2} - \bar{u}\,\dfrac{\tilde{p}_i}{\bar{\rho}} & \bar{u}\,\dfrac{\tilde{p}_i}{\bar{\rho}} + \bar{u} \end{pmatrix} \tag{3.26}$$

where we have used the identities

$$2\bar{u}^2 - \overline{u^2} = \hat{u}^2, \tag{3.27a}$$

$$2\bar{u}^2 + \overline{u^2} = 3\widetilde{u^2}, \qquad \text{and} \tag{3.27b}$$

$$2\bar{u}^3 + \overline{u^3} = 3\bar{u}\,\overline{u^2}, \tag{3.27c}$$

with

$$\hat{u} = \sqrt{u_L\,u_R}, \tag{3.28}$$

the geometric mean, and where

$$\widetilde{a^2} = \tilde{p}_\rho + \bar{p}\,\frac{\tilde{p}_i}{\bar{\rho}^2}. \tag{3.29}$$

3.4. Approximate Eigenvalues and Eigenvectors

Now, the important quantities that are needed for the scheme are the eigenvalues $\tilde{\lambda}_i$ and the eigenvectors $\tilde{\mathbf{e}}_i$ of $\tilde{\mathbf{A}}$, and it is a simple matter to show that these are given by

$$\tilde{\lambda}_{1,2,3} = \bar{u} \pm \tilde{a},\ \bar{u}, \tag{3.30a–c}$$

$$\tilde{\mathbf{e}}_{1,2} = \left(1,\ \bar{u} \pm \tilde{a},\ \frac{\bar{p}}{\bar{\rho}} + \bar{i} + \frac{1}{2}\,\overline{u^2} + \frac{1}{4}\,\frac{\Delta\rho\,\Delta i}{\bar{\rho}} \pm \bar{u}\tilde{a}\right)^{\mathsf{T}}, \qquad \text{and} \tag{3.31a,b}$$

$$\tilde{\mathbf{e}}_3 = \left(1,\ \bar{u},\ \bar{i} + \frac{1}{2}\,\overline{u^2} - \frac{\bar{\rho}\,\tilde{p}_\rho}{\tilde{p}_i} - \frac{1}{4}\,(\Delta u)^2\,\frac{\bar{\rho}}{\tilde{p}_i}\right)^{\mathsf{T}}, \tag{3.31c}$$

where

$$\tilde{a} = \left(\widetilde{a^2} + \frac{1}{4}\,(\Delta u)^2 + \frac{1}{4}\,\Delta\rho\,\Delta i\,\frac{\tilde{p}_i}{\bar{\rho}^2}\right)^{1/2}. \tag{3.32}$$

(We have used the additional identities $\bar{u}^2 - \hat{u}^2 = (1/4)\,(\Delta u)^2$ and $\overline{\rho i} - \bar{\rho}\,\bar{i} = (1/4)\,\Delta\rho\,\Delta i$ in determining these.)

3.5. Projection

It is necessary to project a general jump $\Delta\mathbf{w}$ onto the eigenvectors $\tilde{\mathbf{e}}_i$ as

$$\Delta\mathbf{w} = \sum_{i=1}^{3} \tilde{\alpha}_i\, \tilde{\mathbf{e}}_i, \tag{3.33}$$

and by virtue of equation (3.4), we then have

$$\Delta\mathbf{F} = \sum_{i=1}^{3} \tilde{\lambda}_i\, \tilde{\alpha}_i\, \tilde{\mathbf{e}}_i. \tag{3.34}$$

Solving equation (3.33) gives

$$\tilde{\alpha}_{1,2} = \frac{\Delta p \pm \bar{\rho}\,\tilde{a}\,\Delta u + \frac{1}{4}\,(\Delta u)^2\,\Delta\rho}{2\tilde{a}^2}, \qquad \text{and} \tag{3.35a,b}$$

$$\tilde{\alpha}_3 = \Delta\rho - \frac{\Delta p}{\tilde{a}^2} - \frac{1}{4}\,\frac{(\Delta u)^2\,\Delta\rho}{\tilde{a}^2}. \tag{3.35c}$$

The numerical scheme for solving equation (2.1) using the averages $\tilde{\lambda}_i$, $\tilde{\alpha}_i$, and $\tilde{\mathbf{e}}_i$ follows that in [1]. The resulting scheme is second order accurate and entropy satisfying.

3.6. Approximations for p_ρ and p_i

Finally, it is necessary to give approximations $\tilde{p}_\rho$ and $\tilde{p}_i$ for the derivatives of the equation of state (2.4) that satisfy (3.15). Following the approximations suggested in [2], we propose here similar approximations based on the new averages, i.e.,

$$\widetilde{p_\rho} = p_\rho(\bar{\rho}, \bar{i}), \qquad \text{and} \tag{3.36a}$$

$$\widetilde{p_i} = p_i(\bar{\rho}, \bar{i}), \tag{3.36b}$$

if $\frac{|\Delta\rho|}{\bar{\rho}}$ and $\frac{|\Delta i|}{\bar{i}} \le 10^{-m}$, where m is machine dependent, and

$$\widetilde{p_\rho} = p_\rho(\bar{\rho}, \bar{i}) + \frac{|\Delta\rho|}{|\Delta\rho| + |\Delta i|}\,\frac{\delta p}{\Delta\rho}, \tag{3.36c}$$

$$\widetilde{p_i} = p_i(\bar{\rho}, \bar{i}) + \frac{|\Delta i|}{|\Delta\rho| + |\Delta i|}\,\frac{\delta p}{\Delta i}, \tag{3.36d}$$

otherwise, where

$$\delta p = \Delta p - p_\rho(\bar{\rho}, \bar{i})\,\Delta\rho - p_i(\bar{\rho}, \bar{i}). \tag{3.37}$$

4. IDEAL GAS CASE

In this section, we look briefly at the form for the averages $\widetilde{\lambda_i}$, $\widetilde{\alpha_i}$, and $\widetilde{\mathbf{e}_i}$ given in Section 3 in the case of an ideal gas, where the relevant equation of state is (2.5).

Firstly, since $p = (\gamma - 1)\,\rho\, i$, then $p_\rho = (\gamma - 1)\, i$ and $p_i = (\gamma - 1)\,\rho$, so that the expression (3.37) becomes

$$\delta p = (\gamma - 1)\,\Delta(\rho i) - (\gamma - 1)\,\bar{i}\,\Delta\rho - (\gamma - 1)\,\bar{\rho}\,\Delta i = 0, \tag{4.1}$$

by virtue of (3.7), and thus, the expressions in (3.36a–d) give, in all cases,

$$\widetilde{p_\rho} = (\gamma - 1)\,\bar{i}, \qquad \text{and} \tag{4.2a}$$

$$\widetilde{p_i} = (\gamma - 1)\,\bar{\rho}. \tag{4.2b}$$

Therefore, substituting equation (3.29) into (3.32) and using these averages, after simplification and use of

$$\overline{\rho i} = \bar{\rho}\,\bar{i} + \frac{1}{4}\,\Delta\rho\,\Delta i, \qquad \text{gives} \tag{4.3}$$

$$\tilde{a}^2 = \frac{\gamma\,\bar{p}}{\bar{\rho}} + \frac{1}{4}\,(\Delta u)^2. \tag{4.4}$$

Similarly, the expression $\bar{p}/\bar{\rho} + \bar{i} + \frac{1}{4}\,\Delta\rho\,\Delta i/\bar{\rho}$ in (3.31a–b) becomes $\widetilde{a^2}/(\gamma-1)$ where $\widetilde{a^2} = \gamma\,\bar{p}/\bar{\rho}$, and $\bar{i} - \bar{\rho}\,\tilde{p}_\rho/\tilde{p}_i$ in (3.31c) is zero, leaving $\frac{1}{2}\,\overline{u^2} - \frac{(\Delta u)^2}{4\,(\gamma-1)}$. Finally, in (3.35c) $1 - \frac{1}{4}\frac{(\Delta u)^2}{\tilde{a}^2} = \frac{\gamma\,\bar{p}/\bar{\rho}}{\tilde{a}^2} = \frac{\widetilde{a^2}}{\tilde{a}^2}$.

The corresponding expressions for the averages $\tilde{\lambda}_i$ $\tilde{\alpha}_i$, and $\tilde{\mathbf{e}}_i$ in the case of an ideal gas then coincide with the expressions derived for the Riemann solver [1].

5. TEST PROBLEM AND NUMERICAL RESULTS

The purpose of the one-dimensional test problem of this section is to demonstrate that no loss of accuracy or shock-capturing capability is found when utilising the simplified averages in the Riemann solver based on the scheme of Section 3.

5.1. Test Problem

This test problem is concerned with shock reflection in one dimension of a gas governed by the Euler equations (2.1a–d). We consider a region $0 \le x \le 1$ divided into fifty equally spaced mesh points, and the initial conditions are $(\rho,\, u,\, i,\, p) = (\rho_0,\, u_0,\, i_0,\, p(\rho_0, i_0))$. This represents a gas of constant density and pressure moving towards $x = 0$. The boundary at $x = 0$ is a rigid wall and the exact solution describes shock reflection from the wall. Five equations of state are chosen:

(a) Ideal gas

$$p = (\gamma - 1)\,\rho\, i,$$

where we take the ratio of specific heat capacities γ to be 5/3, corresponding to a monatomic gas;

(b) Stiffened gas (sometimes used for a liquid)

$$p = B\left(\frac{\rho}{\rho_0} - 1\right) + (\gamma - 1)\,\rho\, i,$$

where we take $B = 1$, $\gamma = \frac{5}{3}$;

(c) Copper

$$p = \frac{1}{E + \phi_0}\left[\zeta\,(a_1 + a_2\zeta) + E\,(b_0 + \zeta\,(b_1 + b_2\zeta) + E\,(c_0 + c_1\zeta))\right],$$

where $E = \rho_0\, i$, $\zeta = \frac{\rho}{\rho_o} - 1$, and the constants a_1, a_2, b_0, b_1, b_2, c_0, c_1, and ϕ_0 can be found in [3];

(d) Covolume (used in connection with combustion environments)

$$p = \frac{(\gamma - 1)\,\rho\, i}{1 - \rho\, b},$$

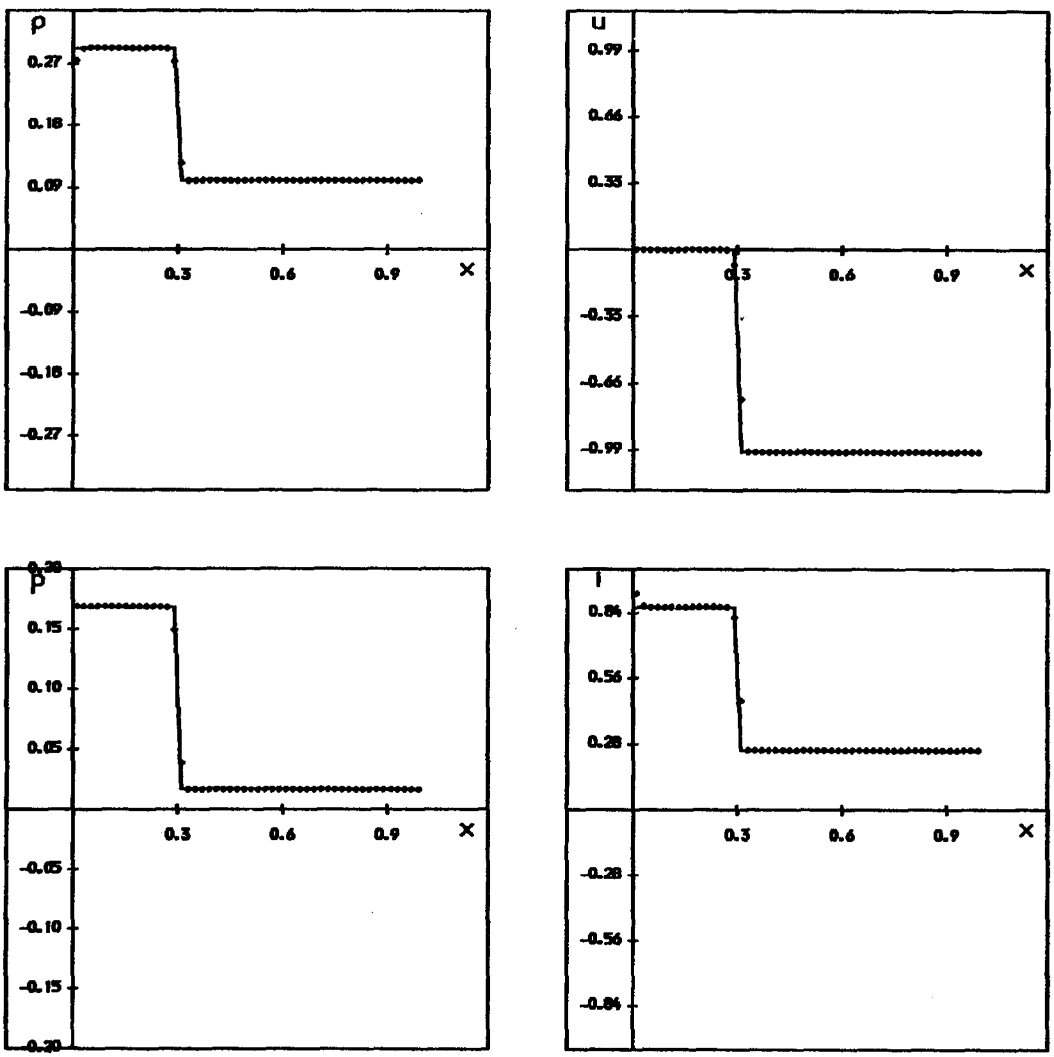
where we take $b = 0.8$, $\gamma = 1.4$; and

(e) Two molecular vibrating gas

$$p + \frac{(\gamma - 1)\,\mu\,\rho}{e^{\rho\,\mu/p} - 1} = (\gamma - 1)\,\rho\, i,$$

where we take $\gamma = 1.4$ and $\mu = 0.5902$, which corresponds to molecular oxygen at 2000 K.

SOLUTION OF THE EULER EQUATIONS
WITH SLAB SYMMETRY
– Shock Reflection



at time $t = 0.575$

Figure 1. Equation of state (a) (ideal gas), with shock strength 10 and reflected boundary conditions at $x = 0$.

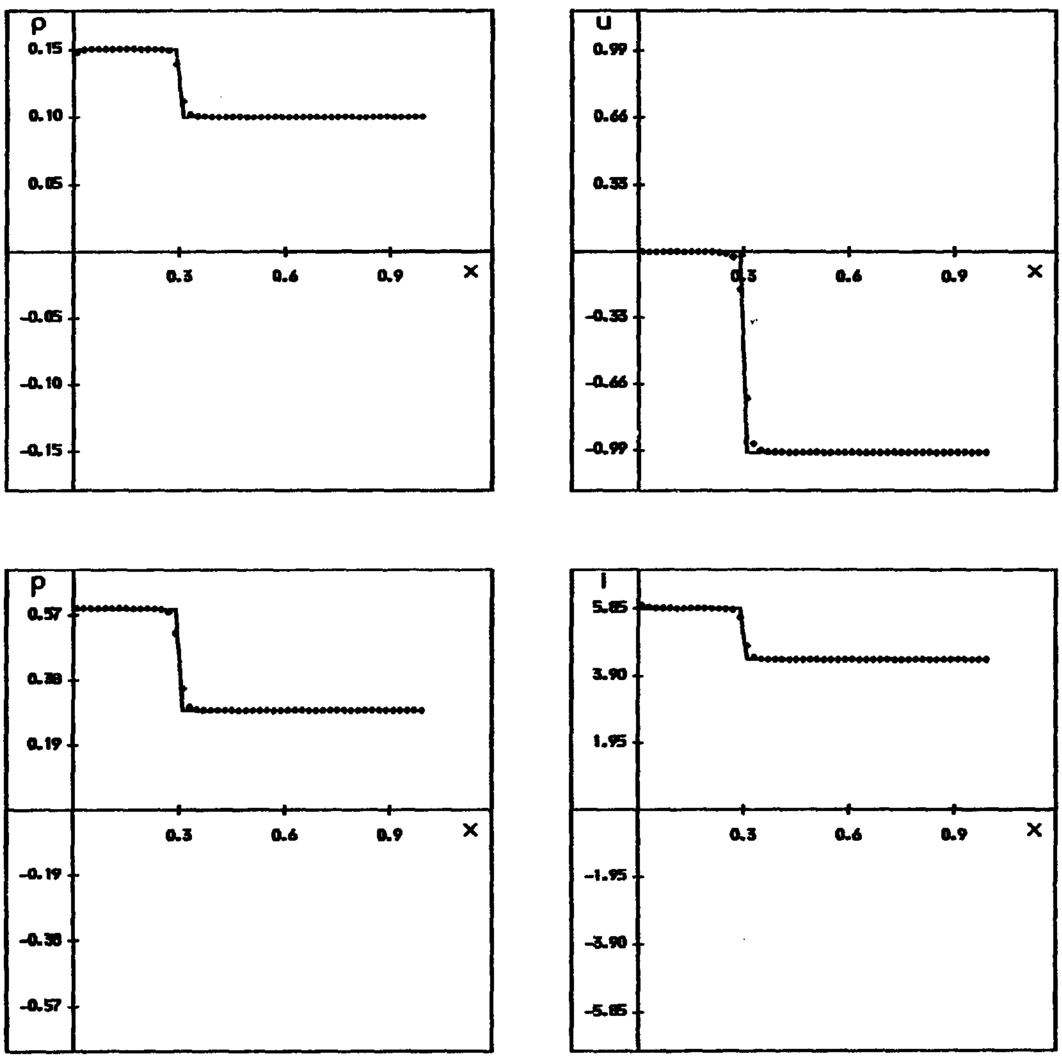
Key: ρ: density; u: velocity; p: pressure; and i: internal energy; —: Exact solution; • • •: Approximate solution.

Parameters: 50 mesh points, 89 time steps, $\Delta x = 0.02$, $\Delta t = 0.0065$; "Superbee" limiter used.

Initial conditions: $\rho_0 = 0.100$; $u_0 = -1.000$, $p_0 = 0.017$, and $i_0 = 0.254$.

SOLUTION OF THE EULER EQUATIONS
WITH SLAB SYMMETRY

– Shock Reflection



at time $t = 0.151$

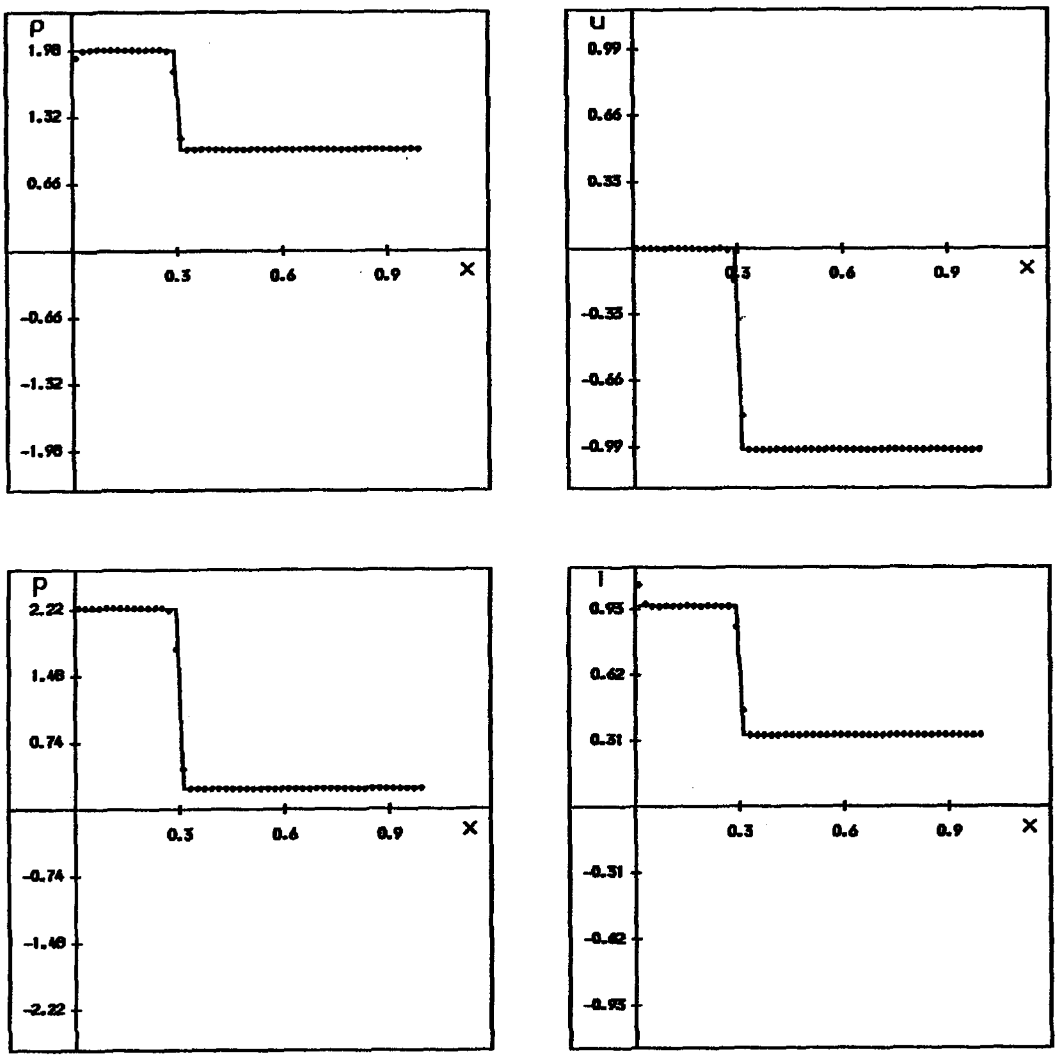
Figure 2. Equation of state (a) (ideal gas), with shock strength 2 and reflected boundary conditions at $x = 0$.

Key: ρ: density; u: velocity; p: pressure; and i: internal energy; —: Exact solution; • • •: Approximate solution.

Parameters: 50 mesh points, 49 time steps, $\Delta x = 0.02$, $\Delta t = 0.0031$; "Superbee" limiter used.

Initial conditions: $\rho_0 = 0.100$; $u_0 = -1.000$, $p_0 = 0.292$, and $i_0 = 4.381$.

SOLUTION OF THE EULER EQUATIONS
WITH SLAB SYMMETRY
– Shock Reflection



at time $t = 0.292$

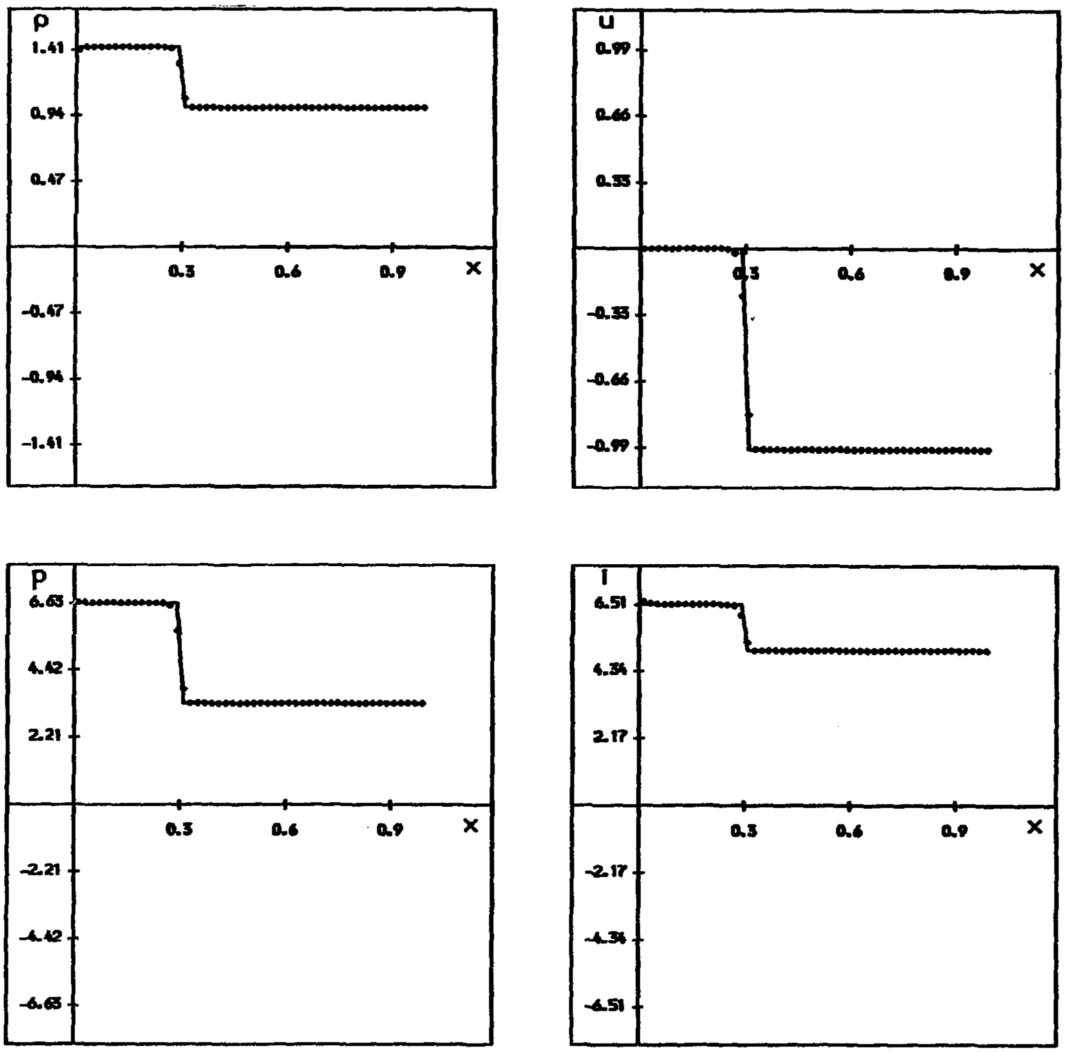
Figure 3. Equation of state (b) (stiffened gas), with shock strength 10 and reflected boundary conditions at $x = 0$.

Key: ρ: density; u: velocity; p: pressure; and i: internal energy; —: Exact solution; • • •: Approximate solution.

Parameters: 50 mesh points, 64 time steps, $\Delta x = 0.02$, $\Delta t = 0.0046$; "Superbee" limiter used.

Initial conditions: $\rho_0 = 1.000$; $u_0 = -1.000$, $p_0 = 0.225$, and $i_0 = 0.337$.

SOLUTION OF THE EULER EQUATIONS
WITH SLAB SYMMETRY
– Shock Reflection



at time $t = 0.128$

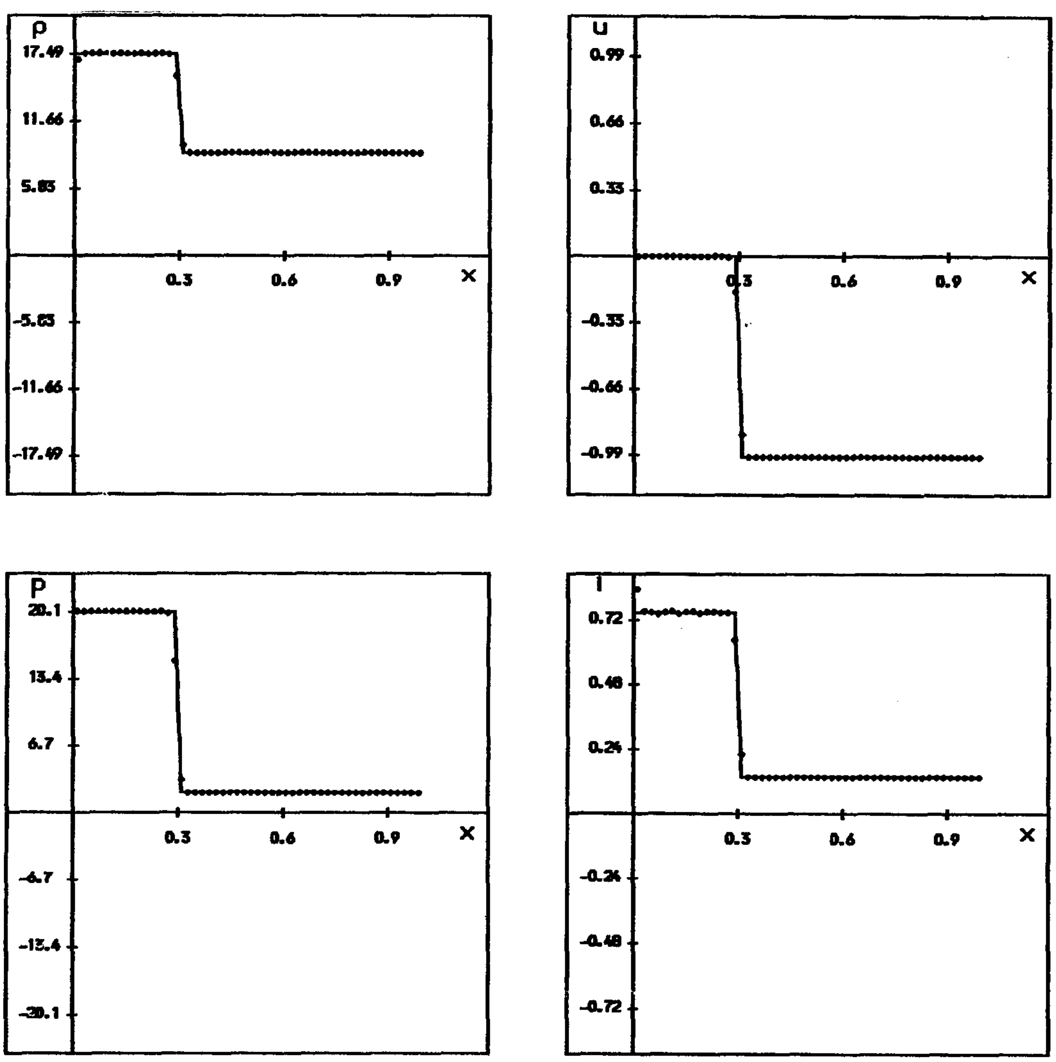
Figure 4. Equation of state (b) (stiffened gas), with shock strength 2 and reflected boundary conditions at $x = 0$.

Key: ρ: density; u: velocity; p: pressure; and i: internal energy; —: Exact solution; • • •: Approximate solution.

Parameters: 50 mesh points, 46 time steps, $\Delta x = 0.02$, $\Delta t = 0.0028$; "Superbee" limiter used.

Initial conditions: $\rho_0 = 1.000$; $u_0 = -1.000$, $p_0 = 3.337$, and $i_0 = 5.005$.

SOLUTION OF THE EULER EQUATIONS
WITH SLAB SYMMETRY
– Shock Reflection



at time $t = 0.285$

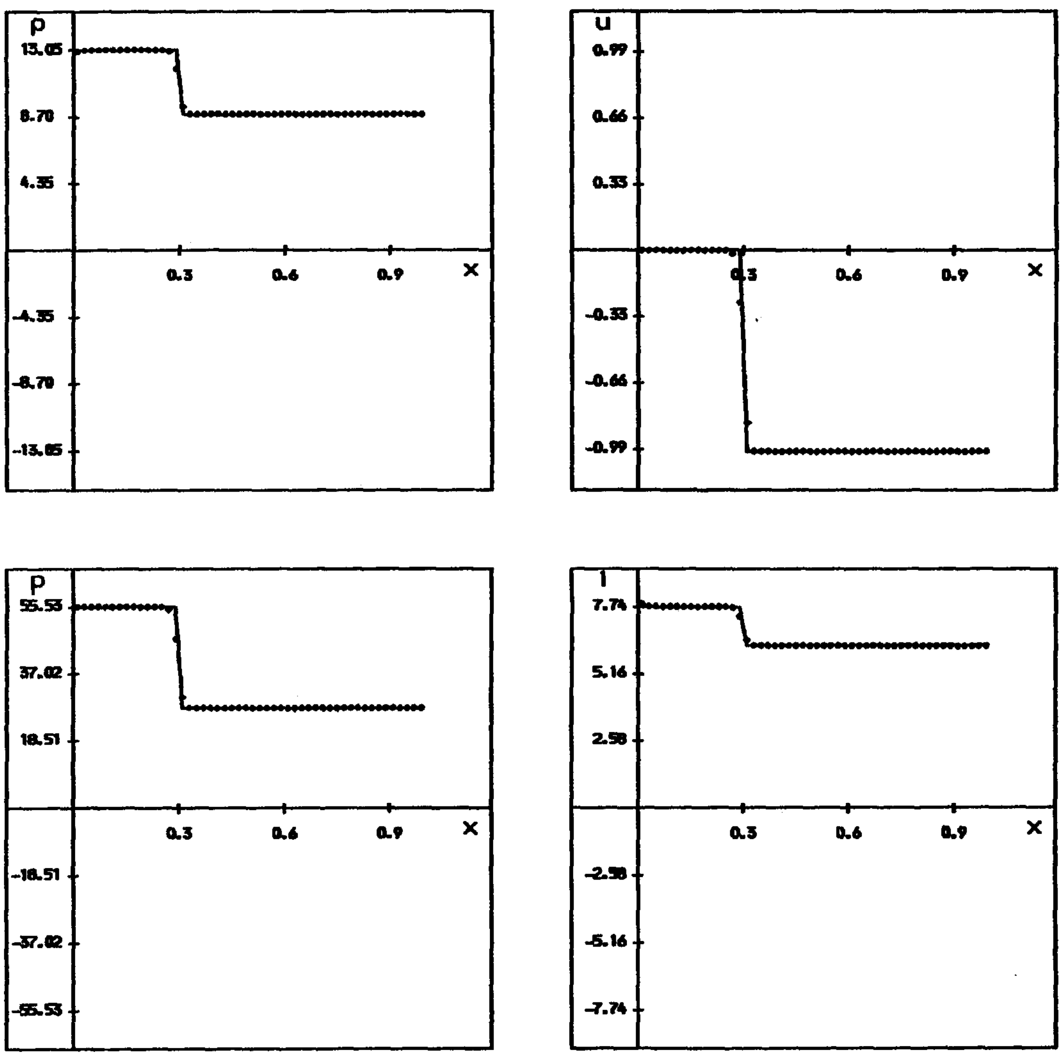
Figure 5. Equation of state (c) (copper), with shock strength 10 and reflected boundary conditions at $x = 0$.

Key: ρ: density; u: velocity; p: pressure; and i: internal energy; —: Exact solution; • • •: Approximate solution.

Parameters: 50 mesh points, 54 time steps, $\Delta x = 0.02$, $\Delta t = 0.0053$; "Superbee" limiter used.

Initial conditions: $\rho_0 = 8.900$; $u_0 = -1.000$, $p_0 = 2.010$, and $i_0 = 0.137$.

SOLUTION OF THE EULER EQUATIONS
WITH SLAB SYMMETRY
– Shock Reflection



at time $t = 0.138$

Figure 6. Equation of state (c) (copper), with shock strength 2 and reflected boundary conditions at $x = 0$.

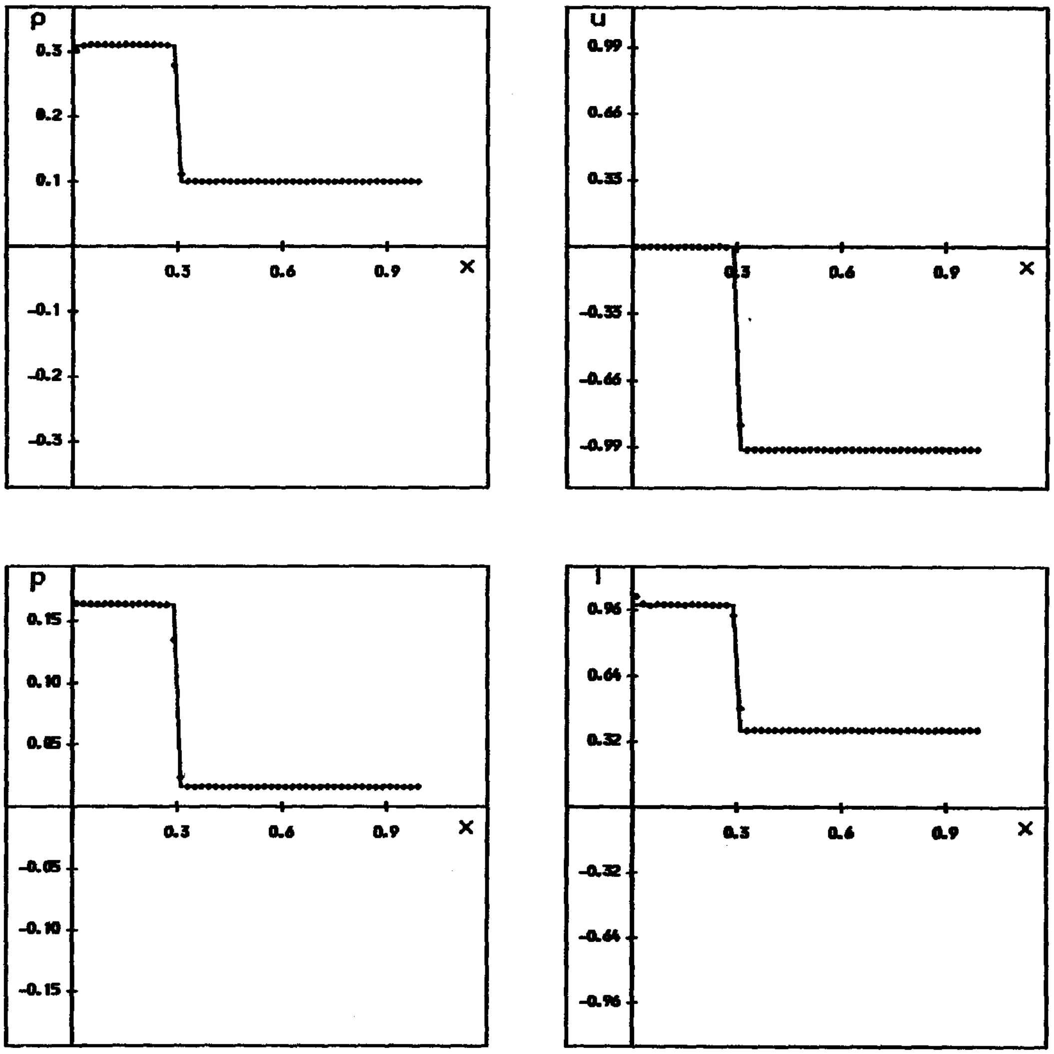
Key: ρ: density; u: velocity; p: pressure; and i: internal energy; —: Exact solution; • • •: Approximate solution.

Parameters: 50 mesh points, 47 time steps, $\Delta x = 0.02$, $\Delta t = 0.0029$; "Superbee" limiter used.

Initial conditions: $\rho_0 = 8.900$; $u_0 = -1.000$, $p_0 = 27.664$, and $i_0 = 6.254$.

SOLUTION OF THE EULER EQUATIONS WITH SLAB SYMMETRY

– Shock Reflection



at time $t = 0.628$

Figure 7. Equation of state (d) (covolume), with shock strength 10 and reflected boundary conditions at $x = 0$.

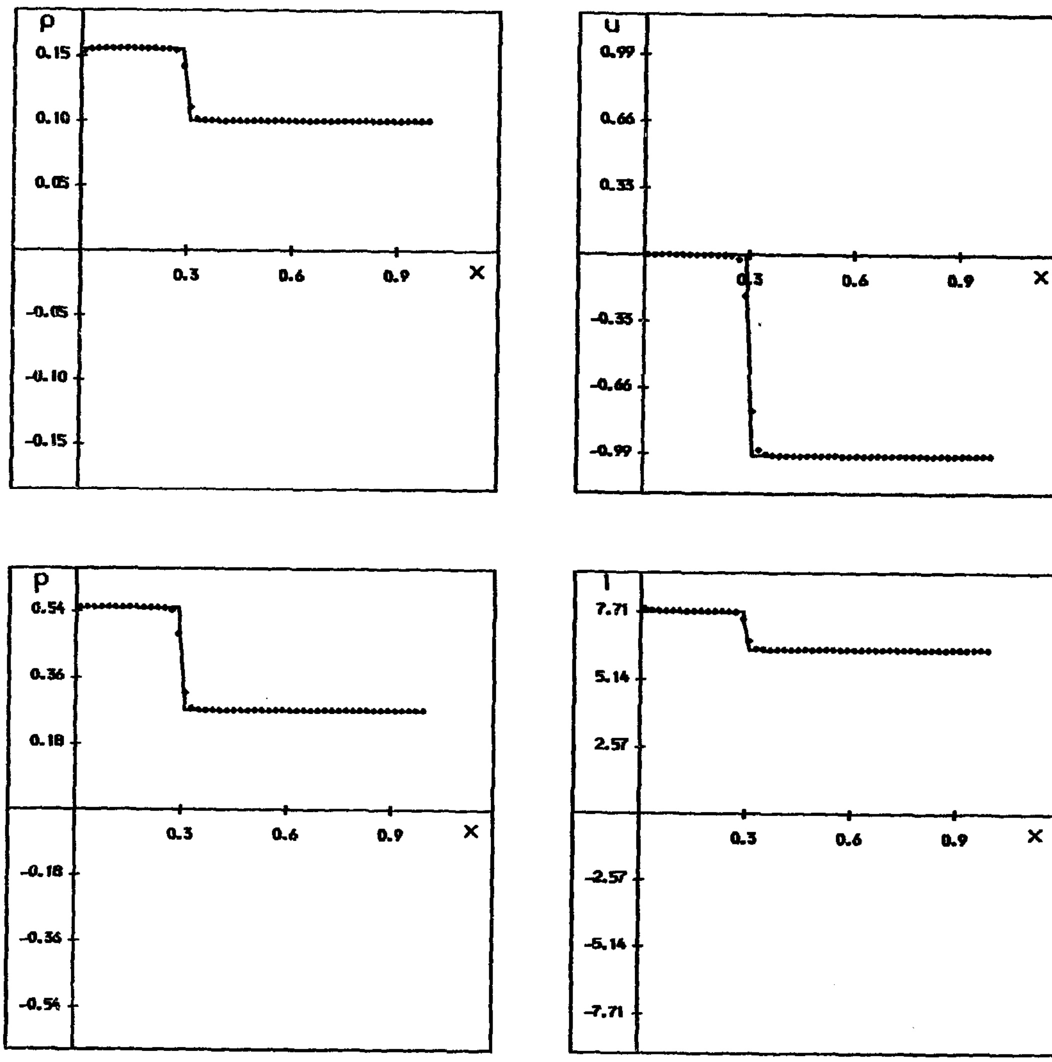
Key: ρ: density; u: velocity; p: pressure; and i: internal energy; —: Exact solution; • • •: Approximate solution.

Parameters: 50 mesh points, 95 time steps, $\Delta x = 0.02$, $\Delta t = 0.0066$; "Superbee" limiter used.

Initial conditions: $\rho_0 = 0.100$; $u_0 = -1.000$, $p_0 = 0.016$, and $i_0 = 0.376$.

SOLUTION OF THE EULER EQUATIONS
WITH SLAB SYMMETRY

– Shock Reflection



at time $t = 0.166$

Figure 8. Equation of state (d) (covolume), with shock strength 2 and reflected boundary conditions at $x = 0$.

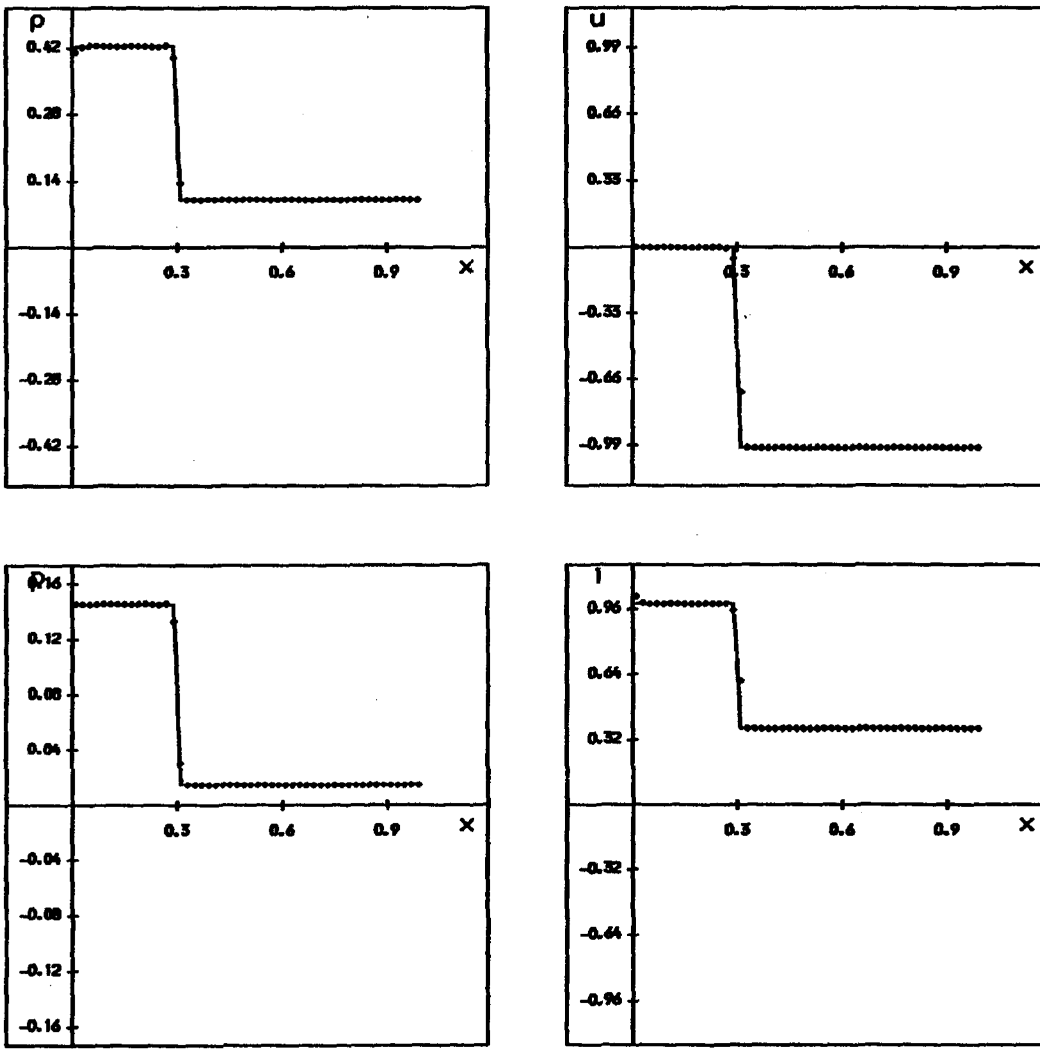
Key: ρ: density; u: velocity; p: pressure; and i: internal energy; —: Exact solution; • • •: Approximate solution.

Parameters: 50 mesh points, 51 time steps, $\Delta x = 0.02$, $\Delta t = 0.0033$; "Superbee" limiter used.

Initial conditions: $\rho_0 = 0.100$; $u_0 = -1.000$, $p_0 = 0.272$, and $i_0 = 6.254$.

SOLUTION OF THE EULER EQUATIONS
WITH SLAB SYMMETRY

- Shock Reflection



at time $t = 0.969$

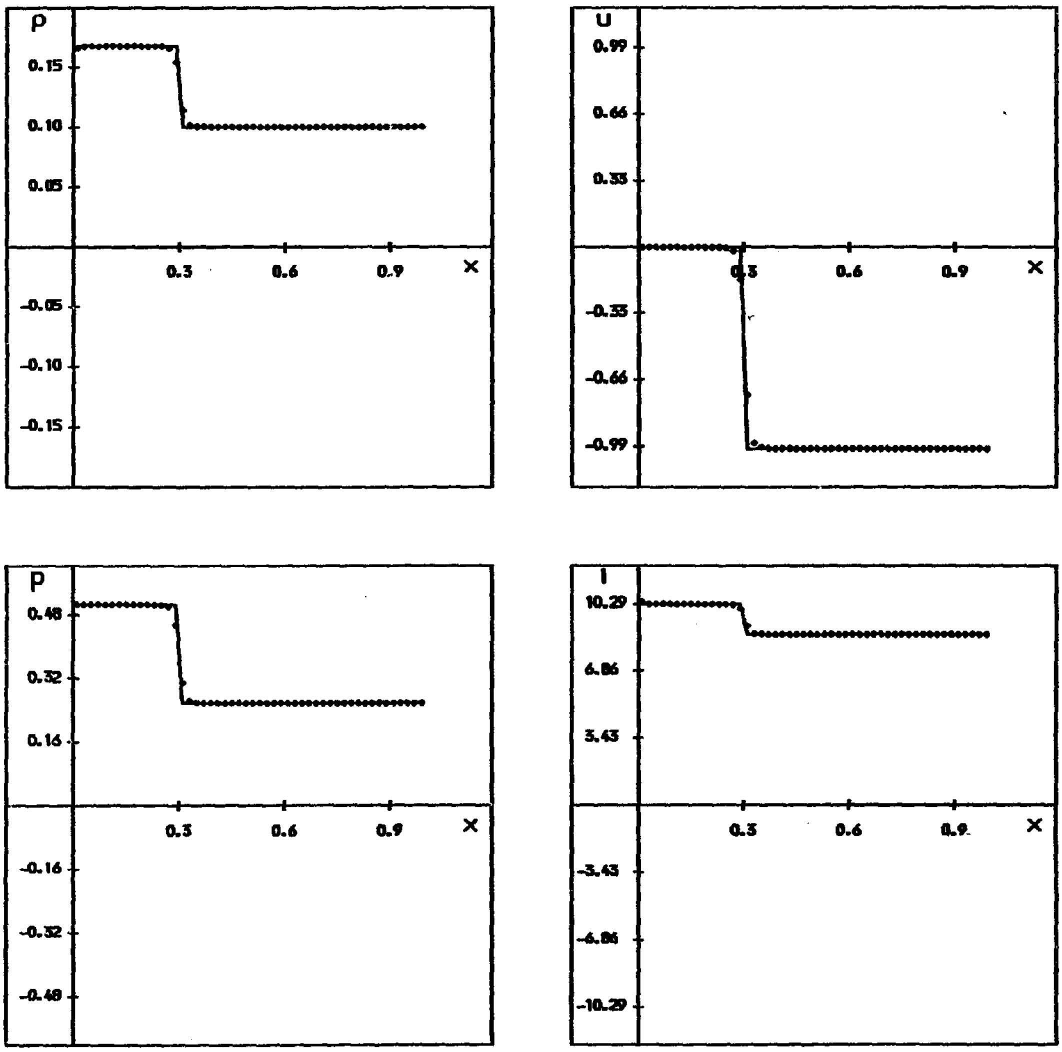
Figure 9. Equation of state (e) (oxygen), with shock strength 10 and reflected boundary conditions at $x = 0$.

Key: ρ: density; u: velocity; p: pressure; and i: internal energy; —: Exact solution; • • •: Approximate solution.

Parameters: 50 mesh points, 142 time steps, $\Delta x = 0.02$, $\Delta t = 0.0068$; "Superbee" limiter used.

Initial conditions: $\rho_0 = 0.100$; $u_0 = -1.000$, $p_0 = 0.015$, and $i_0 = 0.376$.

SOLUTION OF THE EULER EQUATIONS
WITH SLAB SYMMETRY
– Shock Reflection



at time $t = 0.202$

Figure 10. Equation of state (e) (oxygen), with shock strength 2 and reflected boundary conditions at $x = 0$.

Key: ρ: density; u: velocity; p: pressure; and i: internal energy; —: Exact solution; • • •: Approximate solution.

Parameters: 50 mesh points, 59 time steps, $\Delta x = 0.02$, $\Delta t = 0.0034$; "Superbee" limiter used.

Initial conditions: $\rho_0 = 0.100$; $u_0 = -1.000$, $p_0 = 0.258$, and $i_0 = 8.751$.

For each equation of state, we take $u_0 = -1$, and take $\rho_0 = 8.9$ for (c), $\rho_0 = 1$ for (b), and $\rho_0 = 0.1$ for (a), (d), and (e). Two initial conditions are chosen for i_0 (in each case) corresponding to two shock strengths $p_+/p_0 = 10, 2$, where p_+ denotes the pressure behind the shock, and $p_0 = p(\rho_0, i_0)$ denotes the pressure ahead of the shock. The results for these cases are given in Figures 1–10, together with the exact solution when the shock has moved a distance of 0.3. We have used the idea of flux limiters [4] to create a second order algorithm which is oscillation free. The 'superbee' limiter is the one chosen here.

We see that for each equation of state and each of the two shock strengths, the shock speed has been computed accurately, and the shock is captured over only a few cells.

6. CONCLUSIONS

We have extended the Riemann solver presented in [1], for the Euler equations with ideal gases and utilising arithmetic averages of the flow variables, to the case of real gases with a general equation of state, but again using the simplification of arithmetic averages.

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